Implicit derivative

If $z = y \cdot \sin(3x - 1)$ **where** x **and** y **are defined by the system**:

$$\begin{cases} x^2 + y^2 = t(x-1) \\ 2x - 3y = e^t \end{cases}$$

1. **Express** $d^2 z(x, y)$.
2. **Find** $\frac{dz}{dt}$.

Solution

1.
$$d^2z = z''_{xx}(dx)^2 + 2z'_x z'_y dxdy + z''_{yy}(dy)^2$$

Where:

$$z'_x = 3y \cdot \cos(3x-1)$$
$$z''_{xx} = -9y \cdot \sin(3x-1)$$
$$z'_y = \sin(3x-1)$$
$$z''_{yy} = 0$$
$$z'_{xy} = 3 \cdot \cos(3x-1)$$
$$z'_{yx} = 3 \cdot \cos(3x-1)$$
$$d^2z = -9y \cdot \sin(3x-1)\,(dx)^2 + 6 \cdot \cos(3x-1)\,dx\,dy + 0 \cdot (dy)^2$$

2.
$$z'_t = z'_x x'_t + z'_y y'_t$$

Solve the system of implicit equations

$$\begin{cases} x^2 + y^2 - tx + t = 0 \\ 2x - 3y - e^t = 0 \end{cases}$$

$$\begin{cases} (2x-t)dx + 2ydy + (-x+1)dt = 0 \\ 2dx - 3dy - e^t dt = 0 \end{cases}$$

$$\begin{cases} (2x-t)dx/dt + 2ydy/dt + (-x+1) = 0 \\ 2dx/dt - 3dy/dt - e^t = 0 \end{cases}$$

$$\begin{cases} (2x-t)dx/dt + 2ydy/dt = x-1 \\ 2dx/dt - 3dy/dt = e^t \end{cases}$$

Write in matrix form:

$$\begin{bmatrix} 2x-t & 2y \\ 2 & -3 \end{bmatrix} \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \begin{bmatrix} x-1 \\ e^t \end{bmatrix}$$

Solve:

$$dx/dt = \frac{\begin{vmatrix} x-1 & 2y \\ e^t & -3 \end{vmatrix}}{\begin{vmatrix} 2x-t & 2y \\ 2 & -3 \end{vmatrix}} = \frac{-3x+3-2ye^t}{-6x+3t-4y}$$

$$dy/dt = \frac{\begin{vmatrix} 2x-t & x-1 \\ 2 & e^t \end{vmatrix}}{\begin{vmatrix} 2x-t & 2y \\ 2 & -3 \end{vmatrix}} = \frac{2xe^t - te^t - 2x + 2}{-6x+3t-4y}$$

$$z'_t = \frac{-3x+3-2ye^t}{-6x+3t-4y} 3y \cdot \cos(3x-1) + \frac{2xe^t - te^t - 2x + 2}{-6x+3t-4y} \sin(3x-1)$$

$$z'_t = \frac{1}{-6x+3t-4y}\left[(-3x+3-2ye^t)3y\cos(3x-1) + (2xe^t - te^t - 2x + 2)\sin(3x-1)\right]$$